Algebra for Middle School Teachers

(sample pages from chapter 5)

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CHAPTER 5

Algebraic Modeling in Geometry: The Pythagorean Theorem and More

In this chapter, we will explore a variety of interesting geometric and algebraic questions, and in the process we will uncover many important and fascinating aspects of the Pythagorean Theorem, its converse and related results. Algebraic methods will be an important and prominent tool throughout our work. First, a story about my favorite carpenter.

Daryl the Carpenter: There is a general homeowner's rule concerning the hiring (and re-hiring) of skilled individuals to work on your home. In basic terms it is: once you find an honest, skilled and dependable worker (electrician, plumber, carpenter, painter, exorcist, etc.), treat them right, hire them often, and never complain about their work.

Daryl is my carpenter and what a great carpenter he is. I have recommended his services to many of my friends, but this has turned out to be a really big mistake. My friends have now discovered how skilled he is at his job, so consequently it is nearly impossible to find an open time to hire him.

Anyway, what does this story have to do with mathematics? Daryl was installing a skylight in our kitchen, and this involved cutting rectangular holes in my roof and in my kitchen ceiling. He also had to build a chute through the attic connecting the roof to the ceiling. As I watched him prepare for the cutting, I noticed he marked out the area to be cut with a straight edge and a square (right angle ruler). He then commenced to perform several additional measurements, and I asked him what purpose the additional measurements served. I assumed he was using some basic geometry to check whether his figure was actually a rectangle, but I was curious what his response would be. He looked down at me from his ladder and after a bit of chuckling he said, "Cutting a hole in a man's roof is serious business." Since Daryl is a man of " few words," I decided not to press him any further on the geometric rationale behind his measurements.

Skylight in Winter

5.1 The significance of Daryl's measurements and related geometry

1. He first measured the diagonals of the penciled-out quadrilateral (which looked a lot like a parallelogram). Why would this information be useful to Daryl?

2. He also measured the lengths of the sides of a triangle formed by a diagonal and two adjacent sides of the parallelogram, and scratched out some computation on a crumpled coffee stained pad of paper. Again, how would these actions be helpful to Daryl in double checking his work before cutting?

Carpenters' Geometry: Some geometric rationale behind Daryl's measurements of the diagonals

Using ideas from Euclidean Geometry, establish the validity of the following statements, and indicate how some of these might be useful in carpentry. In particular, notice that Exercise 5 (Exercises 5.1) provides a reasonable answer to Question 1 above. In §5.3, we will consider some other mathematics that will help us understand Daryl's further measurements and computations in Question 2 above.

Exercises 5.1:

1. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. Is the converse of this statement valid? If so, prove it and if not, provide a counterexample.

2. If the opposite sides of a quadrilateral are equal in length, then the quadrilateral is a parallelogram.

3. If at least one angle of a parallelogram is a right angle, then the parallelogram is a rectangle.

4. If a quadrilateral is a parallelogram, then its opposite angles have the same measure and its adjacent angles are supplementary.

5. If the diagonals of a parallelogram have equal length, then the parallelogram is a rectangle (cf., Question 1, § 5.1). Give an example to demonstrate that the parallelogram hypothesis cannot be replaced with a quadrilateral hypothesis.

5.2 Classroom Connections: Group Investigations

In this section we will work through part of an 8^{th} grade unit from Book 3 of MathThematics (pages 294 – 296, and 298) focusing on the Pythagorean Theorem. Our goal is not only to explore the mathematical content of these pages, but to show how it relates to other (sometimes more general) mathematical concepts and results (such as some of the previous exercises). By the way, next time I see Daryl, I will be sure to show him Exercise 7 on page 298 of the following classroom lesson!

GOAL

LEARN HOW TO...
- use the Pythagorean theorem to find an unknown side length of a right triangle

AS YOU...
- explore the dimensions of the I.M. Pei Pyramid

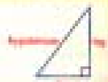
KEY TERMS
- Pythagorean theorem
- hypotenuse
- leg

Exploration 2

The PYTHAGOREAN Theorem

▶ Throughout history, people from many lands have made interesting discoveries about right triangles. One of the most important discoveries is named after the Greek mathematician Pythagoras.

The **Pythagorean theorem** says that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. The **hypotenuse** is the side opposite the right angle. The **legs** are the sides adjacent to the right angle.



9 **Try This as a Class** Use the diagram above.

a. Let *a* and *b* represent the lengths of the legs of the right triangle. Let *c* represent the length of the hypotenuse. Restate the Pythagorean theorem as an equation relating *a*, *b*, and *c*.

b. Suppose you know the length of each leg in a right triangle. How can you find the length of the hypotenuse?

▶ You'll use the Pythagorean theorem to explore the dimensions of a pyramid designed by architect I.M. Pei. The pyramid, shown below, is part of the Louvre Museum in Paris, France.

▶ The Louvre pyramid has a square base and four triangular faces. The *slant height* of the pyramid is the height of one of the triangular faces.

EXAMPLE

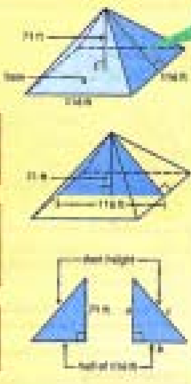
To find the slant height of the Louvre pyramid:

The slant height is the height of one of the triangular faces.

First Imagine slicing through the pyramid to make a two-dimensional triangle.

Then Imagine dividing the two-dimensional triangle into two congruent right triangles.

Now Use the Pythagorean theorem to find the slant height.



10 **Try This as a Class** Use the Example above.

a. Use your equation from Question 9(a) to find the length of the hypotenuse of one of the congruent right triangles.

b. What is the slant height of the Louvre pyramid? Give your answer to the nearest foot.

c. Now that you know the slant height, explain how you can find the area of one triangular face of the pyramid.

▶ You can use the Pythagorean theorem to find the length of one side of a right triangle if you know the lengths of the other two sides. This is shown in the Example on the next page.

EXAMPLE

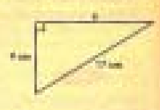
Use the Pythagorean theorem to find the unknown side length of the triangle below.

SAMPLE RESPONSE

The hypotenuse is given, so you need to find the length of one leg.

Let *a* = the unknown side length.

$$a^2 + 9^2 = 17^2$$
$$a^2 + 81 = 289$$
$$a^2 + 81 - 81 = 289 - 81$$
$$a^2 = 208$$
$$\sqrt{a^2} = \sqrt{208}$$
$$a \approx 14.42$$



The length of the unknown side is about 14.4 cm.

Use the Example to answer Questions 11–13.

11 How can you tell from looking at the triangle that the unknown side length is a leg of the triangle, and not the hypotenuse of the triangle?

12 In the equations in the Example, why was 81 subtracted from both sides before the square roots of both sides were found?

13 Why is the length of the third side not an exact measurement?

QUESTION 14

...checks that you can use the Pythagorean theorem to find an unknown side length.

14 CHECKPOINT For each triangle, find the unknown side length.

15 Suppose the length of the longest side of a right triangle is $\sqrt{80}$. The other two side lengths are equal. Give the lengths of the shorter sides. Round your answer to the nearest tenth.

HOMEWORK EXERCISES ▶ See Exs. 8–15 on pp. 218–219.

YOU WILL NEED

For Ex. 14
- graph paper

Section 4 Practice & Application Exercises

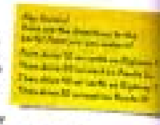
Tell whether a triangle with the given side lengths is acute, right, or obtuse.

1. 5 cm, 12 cm, 13 cm
2. 3 mm, 8 mm, 7 mm
3. 8 in., 10 in., 9 in.
4. 11.3 m, 6.2 m, 7 m
5. 14 cm, 20 cm, 17 cm
6. 15 mm, 12 mm, 9 mm

7. **Carpentry** Carpenters can use a method like the one used by the rope stretchers of ancient Egypt to check whether a corner is "square." For example, a carpenter took the measurements shown to check a right angle on a table. Is the angle opposite the 27 in. diagonal a right angle? Explain your thinking.

For each triangle, find the unknown side length. Give each answer to the nearest tenth.

14. **Visual Thinking** Sharon Ramirez received directions for a party. Use graph paper to sketch Sharon's route. Draw a segment connecting Sharon's house and her friend's house. Find the distance represented by the segment.



15. Can a circular trampoline with a diameter of 16 ft fit through a doorway that is 10 ft high and 8 ft wide? (Assume that the legs of the trampoline can be removed.) Explain your answer.

5.3 Reflections on classroom connections: The Pythagorean Theorem and its converse

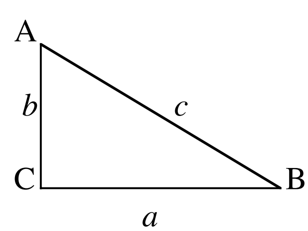
What are some important geometric foundations needed for a middle grade mathematics teacher to understand and effectively teach this lesson? The statement of the Pythagorean Theorem and its applications to calculating certain distances is certainly a minimal requirement, but undoubtedly more background is needed to fully comprehend and utilize this result and related ones in varied situations. For example, Exercises 1 – 7 on page 298 of the MathThematics classroom lesson are not consequences of the Pythagorean Theorem, but follow instead from its converse.

A deeper understanding and appreciation of the mathematics supporting the Pythagorean Theorem and its converse can be reached through the discovery or study of arguments establishing their validity (proof!). In the case of the Pythagorean Theorem, there are known proofs that not only justify the statement of the theorem, but also significantly contribute to a more concrete understanding of the result. Furthermore, the techniques and tools used in one proof of a theorem often can be adapted or modified for use in other arguments (See Exercise 9, Exercises 5.3). We will now consider a few such proofs.

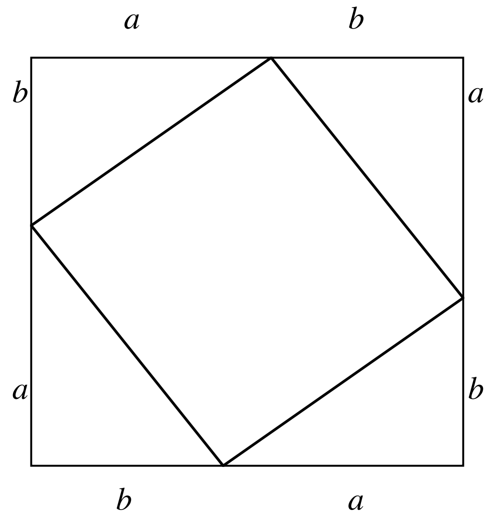
Internet resource: See **www.cut-the-knot.com/pythagoras** for 38 different proofs of the Pythagorean Theorem.

Theorem 5.3.1 (Pythagorean Theorem): If ΔABC is a right triangle, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

Proof 1: Consider the right ΔABC, where $\angle ACB$ is a right angle and sides $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ have lengths AB = c, AC = b, and BC = a.

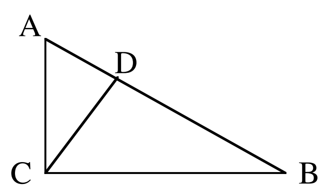


We will prove that $c^2 = a^2 + b^2$. First construct a square of dimensions $a + b$ by $a + b$, and form four right triangles each having a corner of the square as its right angle and with legs of length a and b.



Notice that the hypotenuse of each of the triangles has length c, since each is congruent to ΔABC by SAS. In addition, the quadrilateral inscribed in the square is also a square, since the measure of each of its angles is equal to 180º – (m$\angle$CAB + m$\angle$CBA) = 90º. Hence, $(a + b)^2 = 4(ab/2) + c^2$, and so $a^2 + 2ab + b^2 = 2ab + c^2$. Finally, $a^2 + b^2 = c^2$.

Proof 2: Consider the right ΔABC, where $\angle$ACB is a right angle and sides $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ have lengths AB = c, AC = b, and BC = a.
Construct the perpendicular (altitude) from C to D on side $\overline{AB}$.



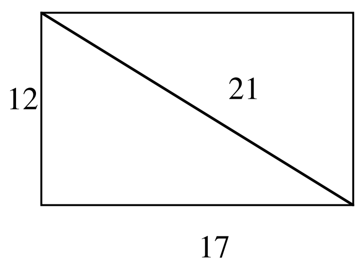
Recall that ΔACD ~ ΔABC ~ ΔBCD (Why?). Hence, $\frac{BC}{BD} = \frac{AB}{BC}$ and $\frac{AC}{AB} = \frac{AD}{AC}$ and so a^2 = (AB)(BD) and b^2 = (AB)(AD). Thus $a^2 + b^2$ = AB(BD + AD) = (AB)(AB) = c^2.

Back to Daryl: If Daryl's penciled-out quadrilateral was actually a rectangle, then we know that his measurements of a diagonal and two adjacent legs would satisfy the Pythagorean Theorem. The problem with this reasoning is that Daryl did not know he had drawn a rectangle, and in fact, this is what he was really trying to prove. So what was his purpose for these measurements and computations? Certainly if his dimensions did not satisfy the Pythagorean Theorem, then he could conclude that the triangle was not a right triangle (and so the parallelogram was not a rectangle). But what if they did satisfy the Pythagorean Theorem -- could he then conclude the triangle was a right triangle? It seems to me that this was Daryl's belief; and lucky for Daryl (and my roof), this is actually a valid deduction. In fact, this is precisely the converse of the Pythagorean Theorem. Here is a short proof of the converse.

Theorem 5.3.2 (Converse of Pythagorean Theorem): If the square of the length of a side of ΔABC equals the sum of the squares of the lengths of the other two sides, then ΔABC is a right triangle with right angle opposite the longest side.

Proof: Assume ΔABC satisfies $a^2 + b^2 = c^2$, where AC = a, BC = b and AB = c. Construct a right ΔEFG with m$\angle$EGF = 90º, EG = a and FG = b. We know that $(EF)^2 = a^2 + b^2 = c^2$, and so EF = c. Hence, $\Delta ABC \approx \Delta EFG$ by SSS, and thus m$\angle$ACB = m$\angle$EGF = 90º.

Back to the classroom lesson: In view of the converse of the Pythagorean Theorem, it is now clear how Daryl's measurements and computations helped him double check his work before cutting. In addition, we know that if his calculations did not obey the Pythagorean Theorem, then the triangle he was checking would not be a right triangle. But then what kind of triangle would it be, and could he determine from his calculations whether it was an acute or obtuse triangle? This is precisely the situation in Exercise 7, page 298 of the classroom lesson. In fact, here is a diagram of that problem.



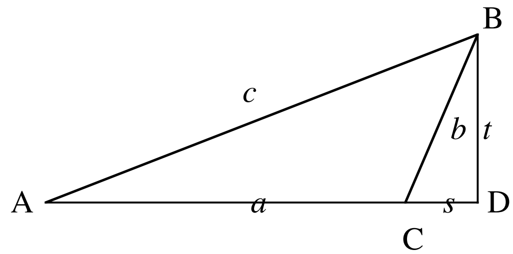
Notice that $12^2 + 17^2 = 433 < 441 = 21^2$, and so the angle opposite the side of length 21 units is not a right angle. If the quadrilateral was drawn to scale, we would probably be able to tell whether that angle was acute or obtuse, but is there a more precise way (not depending on a diagram) to figure this out? What arguments would you make to answer and justify this problem, as well as Exercises 1 – 6 on page 298 of the classroom lesson?

The next proposition addresses this question and establishes a precise delineation of the acute, obtuse and right angle cases. Its proof will not depend on the previous proof of the converse, and this gives us another (more general) validation of the converse.

Theorem 5.3.3 (A generalized Converse of the Pythagorean Theorem): Consider ΔABC, with AC = a, BC = b and AB = c.

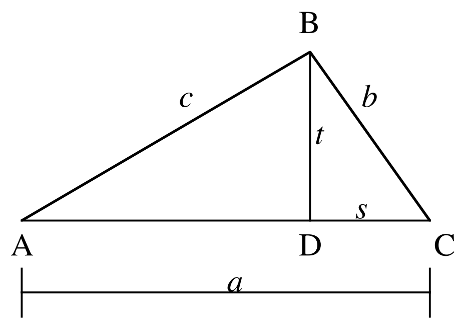
(*i*) If $a^2 + b^2 > c^2$ then the angle opposite c is an acute angle.
(*ii*) If $a^2 + b^2 < c^2$ then the angle opposite c is an obtuse angle.
(*iii*) If $a^2 + b^2 = c^2$ then the angle opposite c is a right angle.

Proof: (*i*) Assume $a^2 + b^2 > c^2$ and suppose that the angle opposite c is not an acute angle. Hence, this angle must be obtuse (Why?). Construct a perpendicular from B to a point D on the extension of $\overline{AC}$, and let CD = s and BD = t.



Note that $b^2 = s^2 + t^2$, and so $[a^2 + (s^2 + t^2)] > c^2 = (a + s)^2 + t^2 =$
$(a^2 + 2as + s^2) + t^2$. Therefore, $0 > 2as$, which is a contradiction.

(*ii*) Assume $a^2 + b^2 < c^2$ and suppose that the angle opposite c is not an obtuse angle. Thus this angle most be acute. Construct a perpendicular from B to a point D on the side $\overline{AC}$, and let CD = s and BD = t.



Using the same approach as in (i), we have $[a^2 + (s^2 + t^2)] < c^2 = (a - s)^2 + t^2 =$
$(a^2 - 2as + s^2) + t^2$, and so $0 < -2as$, which is also a contradiction.

(*iii*) Assume $a^2 + b^2 = c^2$ and suppose the angle opposite c is not a right angle. Then it is either acute or obtuse. Using the ideas in (*i*) and (*ii*), it is routine to complete this part, and we leave this as Exercise 1 (Exercises 5.3).