Chapter 6

Applications of Integration to Area

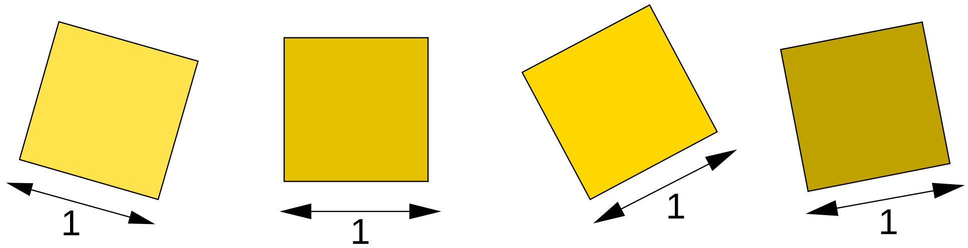
The word *area* is frequently used in real life. For example, the word might be used to describe the size of a given piece of property. How would you informally define the meaning of the word area?

Since ancient times, people have had to devise methods for measuring the areas of plane regions in order to solve real life problems, such as dividing land among heirs. This task turns out to be considerably harder than that of measuring the lengths of plane curves. Why do you think this is so?

Our objective in this chapter is to develop, step by step, starting from scratch, a procedure for measuring the area of a plane region with a continuous boundary and to produce a formula for the area of such a region. So, forget all the formulas related to area that you have learned in the past, and let us go on a discovery adventure!

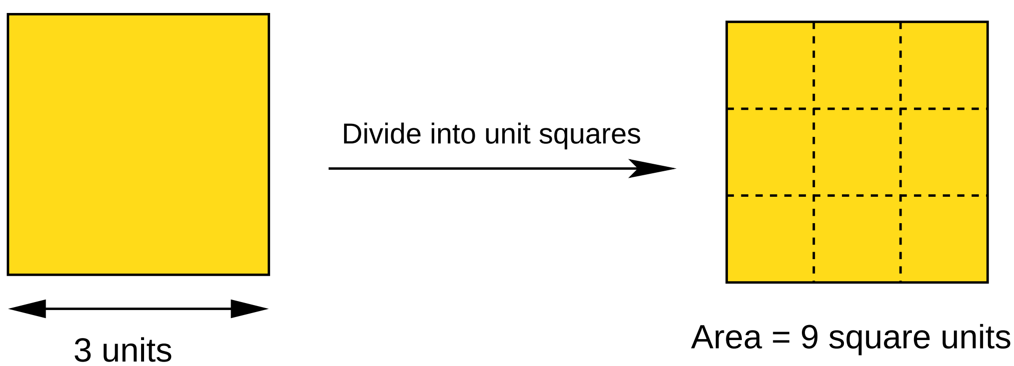
6.1 Areas of polygonal regions

In order to develop an approach for measuring the areas of planar regions, let us start with the simplest geometric figure: a unit square, that is, a square whose sides each have length 1 unit. We define its area to be 1 *square unit*. This definition should make sense to you, and it is unambiguous since all unit squares are congruent.

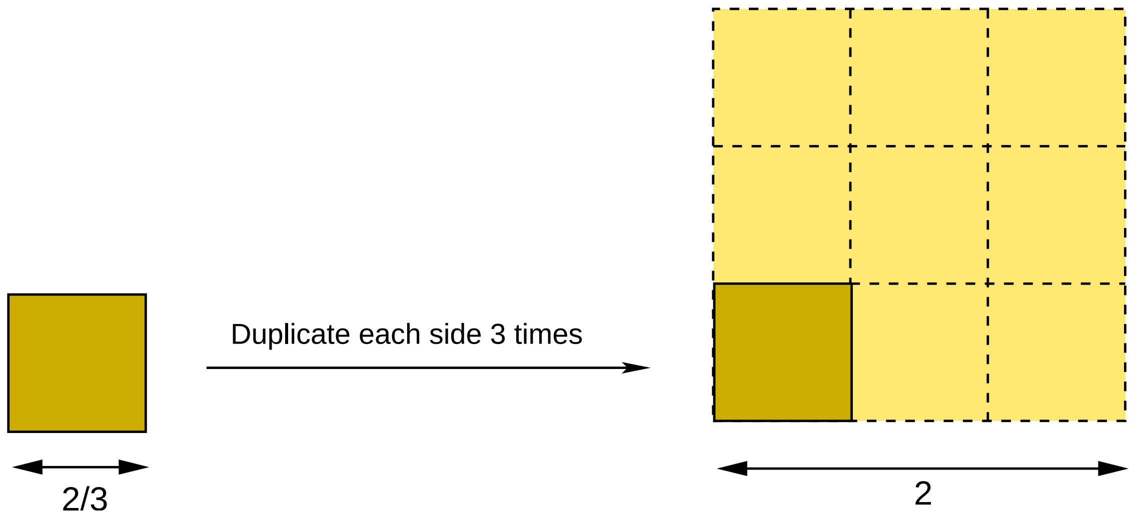


So, from now on, 1 square unit is the area occupied by a unit square. Based on this definition, let us compute the area of a square whose sides each have length a, where a is a positive real number.

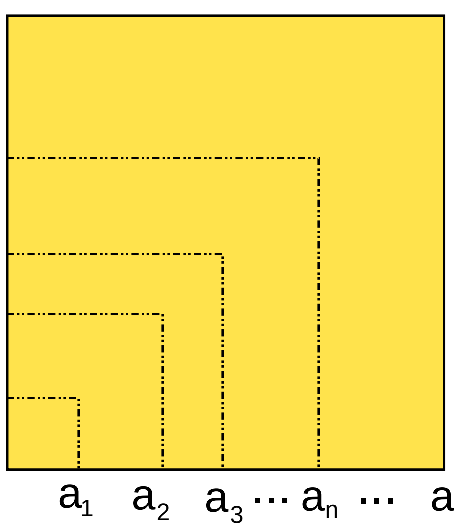
1st case: a is a whole number. (In the figure, $a = 3$.) The a by a square can be divided into a^2 unit squares, and thus its area is a^2 times the area of a unit square, or a^2.



2nd case: a is a rational number that is not a whole number, so $a = p/q$, where p and q are two relatively prime whole numbers with $q \neq 0, 1$. (In the figure, $a = 2/3$.) Remember that, at this point, you possess only a formula for the areas of squares whose side lengths are whole numbers. Therefore, you need to think of a way to reduce this 2nd case to the 1st one. If you duplicate the sides q times, as in the figure below, you will get q^2 squares, each of which is congruent to the original square. Since each side of the resulting large square has length p (a whole number), it follows from the 1st case that the area of the large square is p^2. Therefore, the area of the original square is p^2/q^2, or a^2.



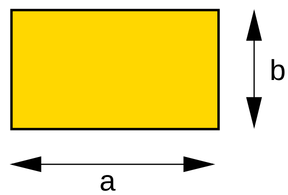
3rd case: a is an irrational number (for example, $a = \sqrt{2}$). The only formula you have so far for computing areas allows you to deal only with squares whose side lengths are rational numbers. Here again, you need to find a way to reduce the current case to the former one. Recall that, since a is a positive irrational number, there exists an increasing sequence $\{a_n\}_n$ of positive rational numbers satisfying $a = \lim_{n \to \infty} a_n$ (for $a = \sqrt{2}$, you may choose the sequence $a_1 = 1.4, a_2 = 1.41, a_3 = 1.414, \ldots$). Consider the squares whose sides are $a_1, a_2, \ldots a_n, \ldots$ (all rational numbers). Their areas are equal to $a_1^2, a_2^2, \ldots a_n^2, \ldots$ respectively, by the 2nd case. Since these squares approach more and more the original square as n increases, you see that the area of the original square must be $\lim_{n \longrightarrow \infty} a_n^2 = a^2$.



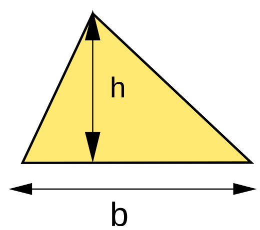
Are you convinced that the area of an a by a square is a^2 for any real number $a > 0$? From now on, you can use this result whenever you need to. To evaluate the areas of random polygonal regions, answer the following questions.

Classroom discussion: Area of a random polygonal region

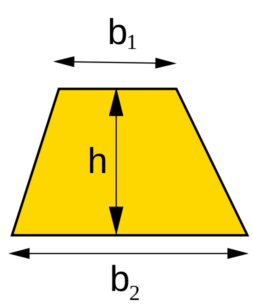
1. What is the area of a rectangle whose length is a and whose width is b? To answer this question, you may proceed either by arguing as in the case of squares or else by considering the square whose sides each have length $a + b$.



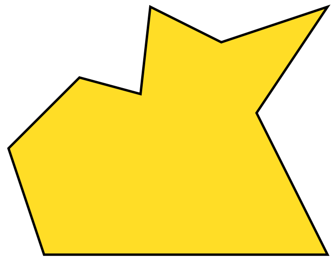
2. What is the area of a triangle whose height is h and whose base has length b? To answer this question, determine first the area of a right triangle using your finding in **1**.



3. What is the area of a trapezoid whose height is h and whose bases have lengths b_1 and b_2? To answer this question, you may decompose the trapezoid into two triangles and use your findings in **2**.



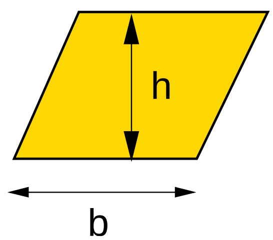
4. Explain how the area of any region that has a polygonal boundary can be computed.



Exercises 6.1

1. Write a paragraph in which you explain to one of your students that the area of a square whose sides each have length π is π^2.

2. The parallelogram below has height h and its base has length b. Using only the area formulas proved so far, find the area of the parallelogram in terms of h and b.

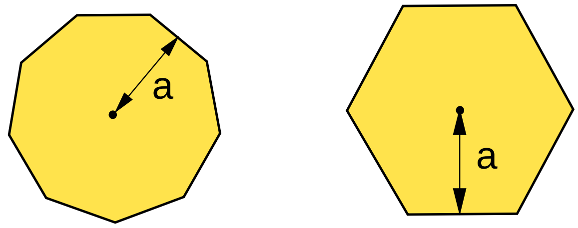


3. Areas of regular polygons

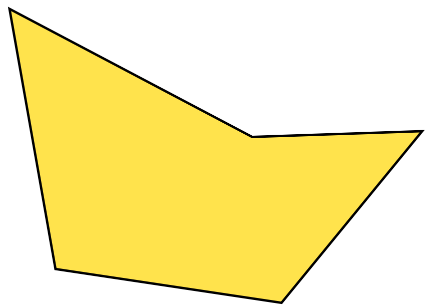
a. A regular polygon has perimeter p and apothem a. Based only on the area formulas proved so far, determine its area in terms of p and a. Recall that the apothem of a regular polygon is the distance between its center and one of its sides.

b. Let n denote the number of sides of the regular polygon in **a**. Using trigonometry, express p in terms of n and a. Then, express the area of the regular n-gon in terms of n and a only.

c. A regular hexagon has apothem a. Express its area in terms of a only.

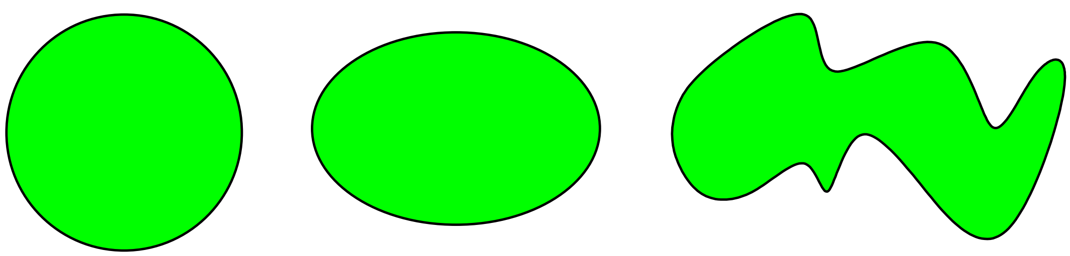


4. Find an approximate value for the area of the polygonal region below using ruler measurements, and any of the area formulas proved so far.



6.2 Approximate values for the area of an irregular shape

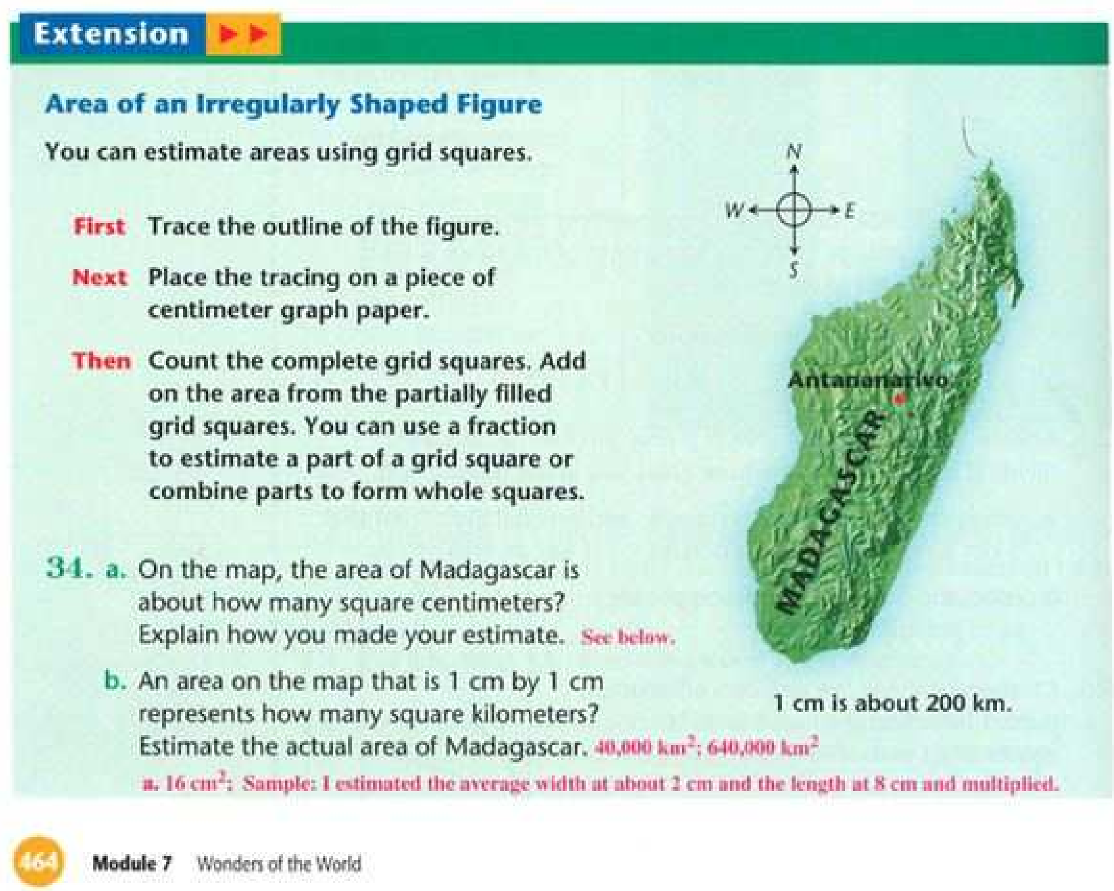
Can you find the areas of the figures below applying the techniques used in §6.1? Explain why or why not.



As you can see from these figures, the boundary of a plane region is *not* necessarily polygonal in general. Throughout this book, a region like the ones above will be referred to as an *irregular* shape. An obvious question now comes to mind: How can we measure the area of an irregular shape?

Classroom connection: Area of Madagascar

The exploration below is taken from the textbook MATH THEMATICS, *Book 1*, p. 464. An approximate value for the area of the island Madagascar is to be found therein. Discuss this exploration in small groups.


Extension ▶▶

Area of an Irregularly Shaped Figure

You can estimate areas using grid squares.

First Trace the outline of the figure.

Next Place the tracing on a piece of centimeter graph paper.

Then Count the complete grid squares. Add on the area from the partially filled grid squares. You can use a fraction to estimate a part of a grid square or combine parts to form whole squares.

34. a. On the map, the area of Madagascar is about how many square centimeters? Explain how you made your estimate. See below.

b. An area on the map that is 1 cm by 1 cm represents how many square kilometers? Estimate the actual area of Madagascar. 40,000 km^2; 640,000 km^2

a. 16 cm^2; Sample: I estimated the average width at about 2 cm and the length at 8 cm and multiplied.

464 **Module 7** Wonders of the World

Classroom connection: Area of a foot

The problem below is taken from the 6^{th} grade textbook CONNECTED MATHEMATICS, *Covering and surrounding*, p. 20. One of the tasks in this problem is to find an approximate value for the area of a foot by counting unit squares and/or fractions of unit squares. Discuss this problem in small groups. In particular, focus on approximating the area of the foot by answering questions **1**-**5** below. Work with the magnified image of the printed foot

1. Find a *lower bound* for the area of the printed foot. A lower bound simply means a specific value that is definitely less than or equal to the exact value of the area of the printed foot.

2. Find an *upper bound* for the area of the printed foot. An upper bound simply means a value that is greater than or equal to the exact value of the area of the printed foot.

3. Alice approximates the area of the printed foot by 131 square units. An approximate value differs from the true value by a certain amount, called the *error*. What is the magnitude of the error generated by Alice's approximation using your answers to **1** and **2**?

4. Bob approximates the area of the printed foot by the average value between the lower and upper bounds found in **1** and **2**. What is the magnitude of the error in this case?

5. If you were to choose between using Alice's approximation or Bob's approximation, which one would you pick? Explain the reasons behind your choice.

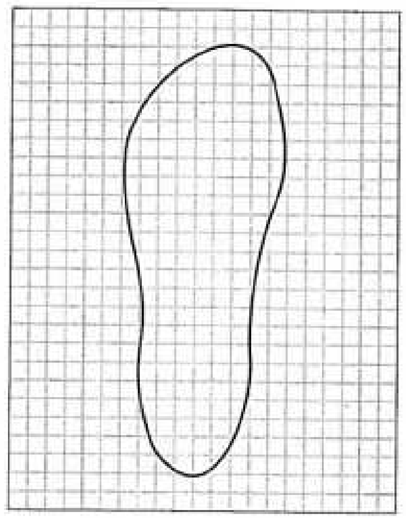
6. What do you think will happen to the error if you use a finer grid?

7. How about if you continue using finer and finer grids? Explain.

Problem 2.1

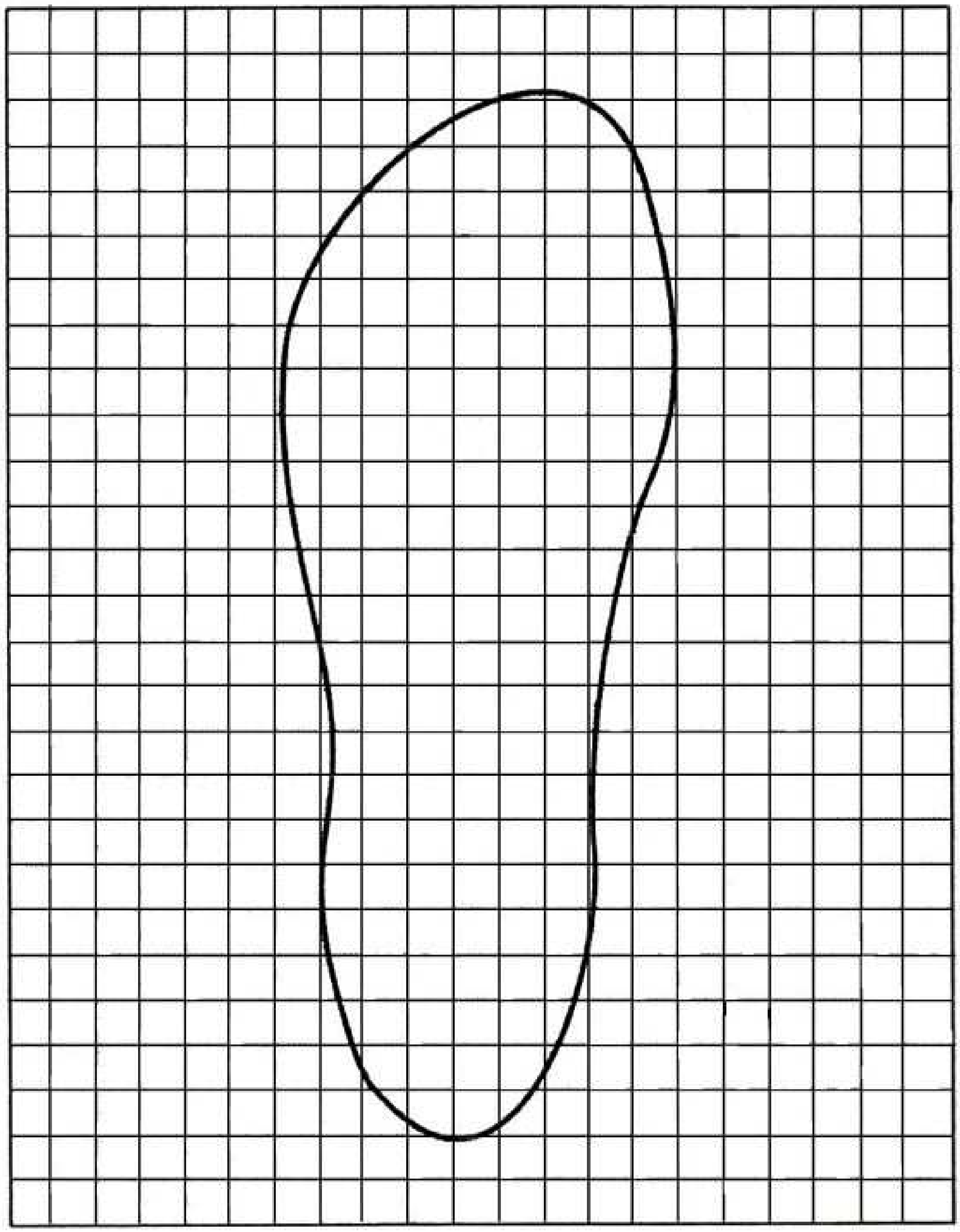
With your group, have a discussion about measuring feet. In what ways can you measure a foot? Which of these measurements would be of interest to shoe companies?

Have each person in your group trace one foot on a piece of grid paper.

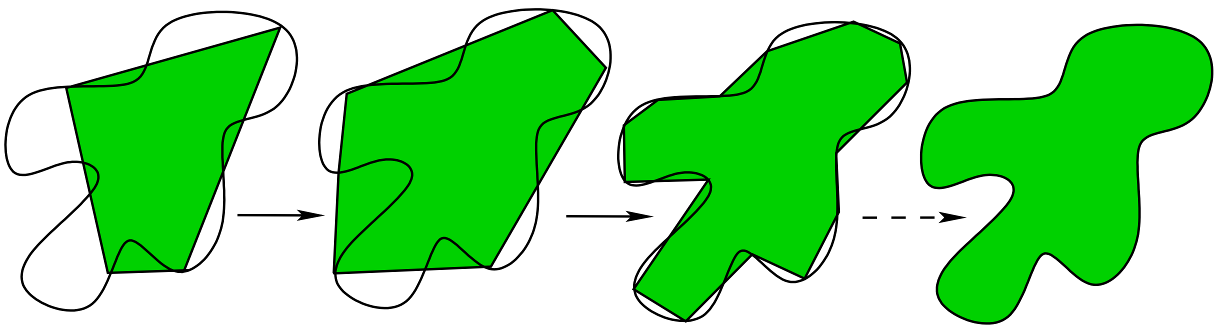


For each person's foot, estimate the length, width, area, perimeter, and any other measures your group thinks should be included. Record your data in a table with these column headings:

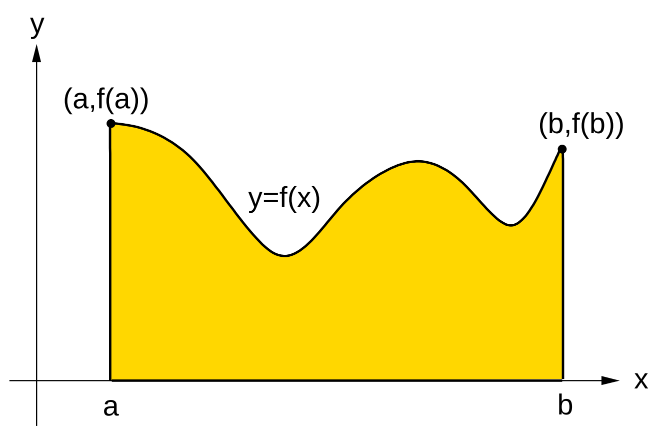
Student	Shoe size	Foot length	Foot width	Foot area	Foot perimeter



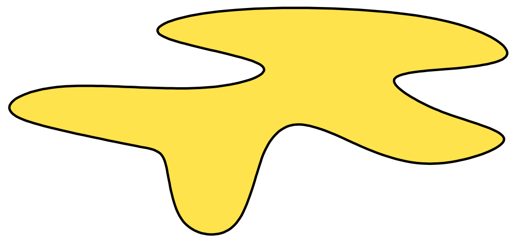
By examining the preceding classroom connections more closely, you can see that the underlying idea in finding approximate values for the area of an irregular shape is to approximate the shape itself by a region whose boundary is polygonal, compute the area of the approximating polygonal region using the results of §6.1, and then use the obtained value to approximate the area of the irregular shape. The smaller the "gaps" between the given irregular shape and the approximating polygonal region, the better your estimate will be for the area. Thus, if the approximating polygonal regions are chosen to approach the given irregular shape more closely, their areas will in turn approach more and more closely the area of the region of interest. The exact value of the area can thus be obtained by passing to the limit:



To illustrate this process more precisely, consider the region of the plane bounded by the graph of a nonnegative continuous function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.



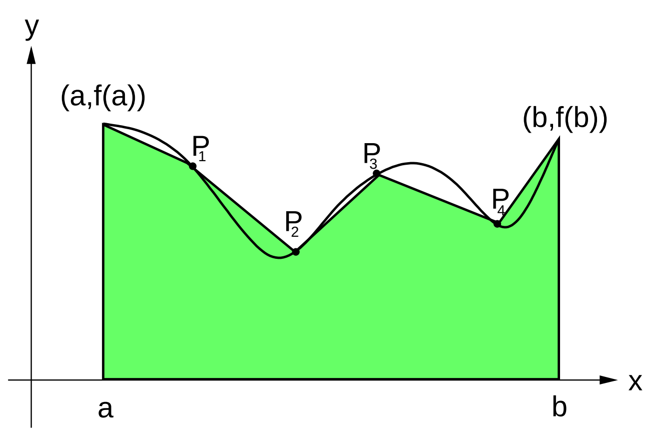
By drawing examples of plane regions, convince yourself that the vast majority of irregular shapes can be always decomposed into subregions, each of which is a region bounded by the graph of some function.



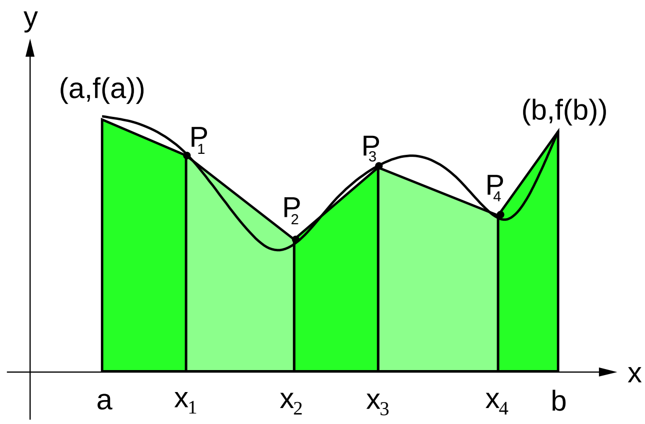
Return to the region below the graph of f and above the x-axis. The only part of the boundary creating a difficulty here is the curved path between points $(a, f(a))$ and $(b, f(b))$. Before looking at the classroom discussion below, take a few minutes to think of a natural way to approximate an irregular shape of this type using polygonal regions.

Classroom discussion: The trapezoidal method

Pick $n-1$ arbitrary points $P_1, P_2, \ldots, P_{n-1}$, on the curved path between the endpoints $(a, f(a))$ and $(b, f(b))$ (in the figure $n = 5$). Then, construct the line segments connecting any two consecutive points lying on this path. The polygonal region thus obtained can serve as an approximating region for the irregular one. Take a few minutes to think about how you can compute the area of this polygonal region.



As you may have already noticed, the polygonal region can be decomposed into n trapezoidal tiles (in the figure, there are 5 such tiles since $n = 5$).



The area of each of these tiles can be computed easily using the formula obtained in §6.1. The result is summarized in the next table.

$$\begin{aligned}
\text{Area of each tile} &= \frac{1}{2}\ \text{width (base 1 + base 2)} \\
\text{Area of Tile 1} &= \frac{1}{2}(x_1 - a)\big(f(a) + f(x_1)\big) \\
\text{Area of Tile 2} &= \frac{1}{2}(x_2 - x_1)\big(f(x_1) + f(x_2)\big) \\
\vdots \qquad & \qquad \vdots \\
\text{Area of Tile } i &= \frac{1}{2}(x_i - x_{i-1})\big(f(x_{i-1}) + f(x_i)\big) \\
\vdots \qquad & \qquad \vdots \\
\text{Area of Tile } n &= \frac{1}{2}(b - x_{n-1})\big(f(x_{n-1}) + f(b)\big)
\end{aligned}$$

The area of the polygonal region denoted by T_n is the sum of the areas of all of these trapezoidal tiles. It is an approximate value for the area $\mathcal{A}$ of the irregular shape.

1. How can you reduce the error that is generated when approximating the area of the irregular shape with T_n? Explain.

2. Louise claims that the exact value of the irregular shape's area can be obtained by taking the limit of T_n as the number of points you select *throughout* the curved path goes to ∞. Do you agree with Louise? Explain.

3. For which functions are the areas T_n necessarily lower bounds for the areas of the resulting irregular shapes? (*Hint:* Think about the shape of their graphs.)
4. For which functions are the areas T_n necessarily upper bounds for the areas of the resulting irregular shapes?

Convenient choice: Originally, you picked points $P_1, P_2, \ldots, P_{n-1}$ along the curved path randomly. However, if you are seeking a simple expression for the approximating areas T_n, it is helpful to select these points so that their x-coordinates $x_1, x_2, \ldots x_{n-1}$ are evenly spaced along the interval $[a, b]$; that is,

$$x_1 - a = x_2 - x_1 = \ldots = b - x_{n-1}.$$

Denote this common value by δ.
5. Show that in this case,

$$T_n = \frac{\delta}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right). \tag{1}$$

6. Find the value of δ in terms of $a, b,$ and n only. Then, replace this value in the expression of T_n above.
7. Express each coordinate x_i in terms of a, b, and i.
8. Write a calculator program that will compute T_n for arbitrary n. Your program should be set up to provide the value of T_n once you enter information about f, a, b, and n.

This method of finding approximate values for the area of the region below the graph of a function and above the x-axis is called **the trapezoidal method**. Formulas of type (1) are attributed to the mathematician Bernhard Riemann and are usually referred to as **Riemann sums for the function f on the interval $[a, b]$.**

Example

Let $\mathcal{A}$ be the area of the plane region bounded by the graph of $f(x) = 1/x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 5$, and let T_n be the area of the polygonal region with n trapezoids of equal width, as described in *Convenient choice* above.
1. Compute by hand T_4 and T_8. Are these approximations upper or lower bounds for $\mathcal{A}$?

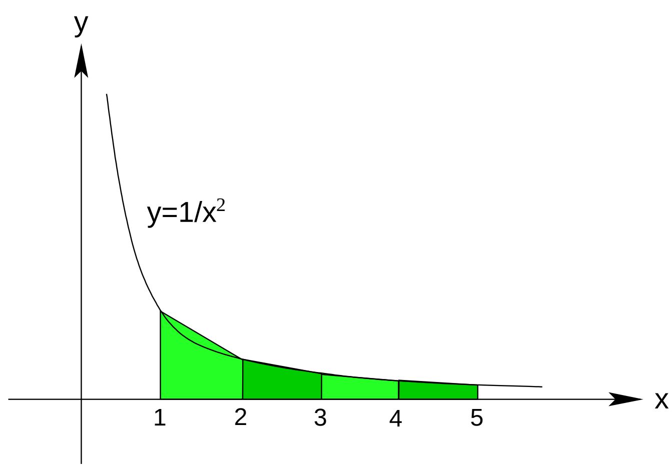
2. Compute $T_{10}, T_{20}, \ldots, T_{200}$ using your calculator. Tabulate your results.

3. Based on your table and on your analysis of what happens as n increases, conjecture an approximate value for $\mathcal{A}$ with an error of magnitude less than 10^{-3}.

Solution

1. The area T_4 is obtained by applying the formula (1) or else by computing directly the area of each of the trapezoidal tiles involved.

$$T_4 = \frac{5-1}{8}\left(1 + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{9} + 2\cdot\frac{1}{16} + \frac{1}{25}\right) \approx 0.94361.$$

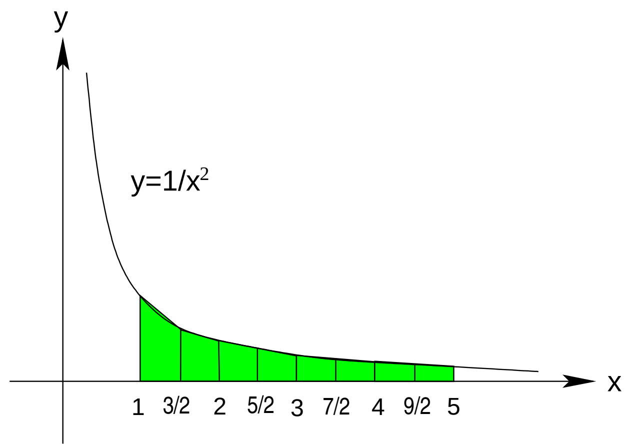


Since the graph of the function is concave upward, T_4 is an upper bound for $\mathcal{A}$; that is, $\mathcal{A} \leq T_4 \approx 0.94361$.

Similarly, the area T_8 may be obtained by applying directly the formula (1) as follows.

$$T_8 = \frac{5-1}{16}\left(1+2\cdot\frac{4}{9}+2\cdot\frac{1}{4}+2\cdot\frac{4}{25}+2\cdot\frac{1}{9}+2\cdot\frac{4}{49}+2\cdot\frac{1}{16}+2\cdot\frac{4}{81}+\frac{1}{25}\right) \approx 0.83953.$$

For the same reason, $T_8 \approx 0.83953$ is also an upper bound for $\mathcal{A}$.



2.

n	T_n
10	0.82568
20	0.80656
30	0.80293
40	0.80165
50	0.80106
60	0.80073
70	0.80054
80	0.80041
90	0.80033
100	0.80026

n	T_n
110	0.80023
120	0.80018
130	0.80016
140	0.80013
150	0.80012
160	0.80010
170	0.80009
180	0.80008
190	0.80007
200	0.80006

3. By analyzing the table above, it seems reasonable to conjecture that the approximation $\mathcal{A} \approx 0.800$ generates an error of magnitude less than 10^{-3}; that is, $.799 < \mathcal{A} < .801$. In §6.3, you will learn how to compute the exact value of the area $\mathcal{A}$, and thus you will have the opportunity to check whether your conjecture is true or false.

Classroom discussion: The rectangular methods

The goal of this classroom discussion is to approximate the region in the plane that is under the graph of a continuous nonnegative function with appropriate *rectangular* tiles instead of trapezoidal tiles in order to find approximate values for its area. Pick $n - 1$